Predicates and Quantifiers

Lecture 9
Section 3.1

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Outline

Predicates

- We can express statement such as “$1 + 2 = 2 + 1$” and “$1 + 3 = 3 + 1$” and “$2 + 3 = 3 + 2$“ and so on, but how do we express the fact that the statement is true for *any* two numbers?
- We need a way to express the idea for all numbers in a single statement.

Predicates

- A predicate is a sentence that contains a finite number of variables, and becomes a statement (true or false) when values are substituted for the variables.
- If the predicate is "In the month of x, there is a good chance of snow at HSC," then the predicate becomes true when x = "December," "January," "February," or "March" and false when x = "April," . . . , "November."
- Predicates may have any number of variables: "If the x is y, then my z will w."

Domains of Predicate Variables

- The domain D of a predicate variable x is the set of all values that x may take on.
- Let $P(x)$ be the predicate.
- Then x is a free variable.
- The truth set of $P(x)$ is the set of all values of $x \in D$ for which $P(x)$ is true.

Domains of Predicate Variables

- In the last example, the domain of x is

$$D = \{\text{January}, \text{February}, \text{March}, \text{April}, \text{May}, \text{June}, \text{July}, \text{August}, \text{September}, \text{October}, \text{November}, \text{December}\}$$

and the truth set of $P(x)$ is

$$\{\text{January}, \text{February}, \text{March}, \text{December}\}.$$

Outline

The Universal Quantifier

- The symbol $\forall$ is the universal quantifier.
- The statement
$$\forall x \in S, P(x)$$
means "for all x in S, $P(x)$," where $S \subseteq D$.
- In this case, x is a bound variable, bound by the quantifier $\forall$.
- The qualified statement $\forall x \in S, P(x)$ is either true or false.

The Universal Quantifier

- The statement

$$\forall x \in S, P(x)$$

  is true if $P(x)$ is true for *all* x in S.
- The statement is false if $P(x)$ is false for *at least one* x in S.

Examples

- The statement “7 is a prime number” is true.
- The predicate “x is a prime number” is neither true nor false.
- The statement “$\forall x \in \{2, 3, 5, 7\}$, x is a prime number” is true.
- The statement “$\forall x \in \{2, 3, 6, 7\}$, x is a prime number” is false.

Examples of Universal Statements

- $\forall x \in \{1, 2, \ldots, 10\}, x^2 > 0.$
- $\forall x \in \{1, 2, \ldots, 10\}, x^2 > 100.$
- $\forall x \in \mathbb{R}, x^3 - x \geq 0.$
- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + xy + y^2 \geq 0.$
- $\forall x \in \varnothing, x^2 > 100.$

Algebraic Identities

- Algebraic identities are universal statements.
- When we write the algebraic identity

$$(x + 1)^2 = x^2 + 2x + 1$$

  what we mean is

$$\forall x \in \mathbb{R}, (x + 1)^2 = x^2 + 2x + 1.$$

Outline

The Existential Quantifier

- The symbol $\exists$ is the existential quantifier.
- The statement
$$\exists x \in S, P(x)$$
means "there exists x in S such that $P(x)$," where $S \in D$.
- x is a bound variable, bound by the quantifier $\exists$.
- The qualified statement $\exists x \in S, P(x)$ is either true or false.

The Existential Quantifier

- The statement
$$\exists x \in S, P(x)$$
is true if $P(x)$ is true for *at least one* x in S.
- The statement is false if $P(x)$ is false for *all* x in S.

Examples of Existential Statements

- $\exists x \in \{1, 2, \ldots, 10\}, x^2 > 0$.
- $\exists x \in \{1, 2, \ldots, 10\}, x^2 > 100$.
- $\exists x \in \mathbb{R}, x^3 - x \geq 0$.
- $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + xy + y^2 \geq 0$.
- $\exists x \in \varnothing, x^2 > 100$.

Algebraic Solutions

- An algebraic equation has a solution if there exists a value for x that makes it true.

$$\exists x \in \mathbb{R}, (x+1)^2 = x^2 + x + 2.$$

The Universal and Existential Quantifiers

- Is it true that

$$\forall x \in D, P(x) \rightarrow \exists x \in D, P(x).$$

The Universal and Existential Quantifiers

- Is it true that
$$\forall x \in D, P(x) \rightarrow \exists x \in D, P(x).$$
- How can we modify the statement to make it true?

Boolean Identities

- A boolean expression
  - Is not a contradiction if there exist truth values for its variables that make it true.
  - Is not a tautology if there exists truth values for its variables that make if false.
- The expression

$$p \wedge q \leftrightarrow p \vee q$$

is neither a contradiction nor a tautology.

Outline

Universal Conditional Statements

- The statement

$$\forall x \in S, P(x) \to Q(x)$$

  is a universal conditional statement.
- Example: $\forall x \in \mathbb{R}$, if $x > 0$, then $x + \frac{1}{x} \geq 2$.
- Example: $\forall n \in \mathbb{N}$, if $n \geq 15$, then $n^2 + 12n - 16 > 32n - 92$.

Universal Conditional Statements

- Suppose that predicates $P(x)$ and $Q(x)$ have the same domain D.
- If $P(x) \to Q(x)$ for all x in the truth set of $P(x)$, then we write

$$P(x) \Rightarrow Q(x).$$

Examples

- Examples
  - $x > 0 \Rightarrow x + \frac{1}{x} \geq 2$.
  - n is a multiple of 6 $\Rightarrow$ n is a multiple of 2 and n is a multiple of 3.
  - n is a multiple of 8 $\Rightarrow$ n is a multiple of 2 and n is a multiple of 4.
- Are these statements true if we replace $\Rightarrow$ with $\Leftarrow$?

Universal Biconditional Statements

- Suppose that predicates $P(x)$ and $Q(x)$ have the same domain D.
- If $P(x) \leftrightarrow Q(x)$ for all x in the truth set of $P(x)$, then we write

$$P(x) \Leftrightarrow Q(x).$$

Examples

- Which of the following are true?
  - $x > 0 \Leftrightarrow x + \frac{1}{x} \geq 2$.
  - n is a multiple of 6 $\Leftrightarrow$ n is a multiple of 2 and n is a multiple of 3.
  - n is a multiple of 8 $\Leftrightarrow$ n is a multiple of 2 and n is a multiple of 4.

Solving Algebraic Equations

- Steps in solving algebraic equations are equivalent to universal conditional statements.
- The step

$$x + 3 = 8$$
$$\therefore x = 5$$

is justified by the statement

$$x + 3 = 8 \Rightarrow x = 5.$$

Solving Algebraic Equations

- An algebraic step from $P(x)$ to $Q(x)$ is reversible only if $P(x) \Leftrightarrow Q(x)$.
- Reversible steps
  - Adding 1 to both sides.
  - Taking logarithms.
- Not reversible steps
  - Squaring both sides.

Solving Algebraic Equations

- Correct:
  - The solution to $x - 1 = 3$ is $x = 4$.
  - The solution to $2^x = 100$ is $x = \frac{\log 100}{\log 2}$.
- Incorrect:
  - The solution to $x + \sqrt{x} = 6$ is $x = 4$ or $x = 9$.

Algebra Puzzler

- Find the error(s) in the following "solution."

$$
\begin{aligned}
\frac{x}{x+2} &\geq 3 \\
\frac{x+2}{x} &\leq \frac{1}{3} \\
1 + \frac{2}{x} &\leq \frac{1}{3} \\
\frac{2}{x} &\leq -\frac{2}{3} \\
\frac{1}{x} &\leq -\frac{1}{3} \\
x &\geq -3.
\end{aligned}
$$

Outline

Assignment

Assignment

- Read Section 3.1, pages 96 - 105.
- Exercises 3, 4, 6, 11, 12, 14, 18, 19, 25, 29, 32, page 106.